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Asymptotic solution of the first boundary-value problem of the theory of elasticity of the forced vibrations of an isotropic strip ☆

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ABSTRACT

The first boundary-value problem of the theory of elasticity of the forced vibrations of an isotropic strip is solved by an asymptotic method. The asymptotic form of the components of the stress tensor and the displacement vector, which differ in principle from the asymptotic form in the corresponding static problem, is established. All the required quantities in the inner problem are determined and the conditions for resonance to occur are established. The solution in the dynamic boundary layer is constructed and the fundamental (inner) and boundary solutions are matched.

The plane static first boundary-value problem for a rectangular region was solved in Ref. 1 by an asymptotic method (the values of the corresponding components of the stress tensor were specified on the longitudinal edges of the rectangle). A relation was established between the solution obtained and the classical theory of the bending and stretching/compressing of beams and rods, and also with the Saint-Venant principle.[1,2] The first boundary-value problem for thin bodies like plates and shells was considered in Refs 2,3 using the equations of the three-dimensional problem of the theory of elasticity. The asymptotic method turned out to be particularly effective for solving non-classical boundary-value problems, i.e., when the conditions of the second or mixed boundary-value problems of the theory of elasticity[2,4] are specified on the end surfaces of thin bodies. A class of non-classical dynamic boundary-value problems on natural and forced vibrations was solved in Refs 5–9. A review of papers on the subject can be found in Ref. 10.

Below we will show that the asymptotic method is also effective for solving the dynamic first boundary-value problem for thin bodies.

1. Fundamental equations and boundary conditions and the solution of the inner problem

The problem is as follows: it is required to obtain, in a rectangular region (a strip)

$$D = \{(x, y) : x \in [0, l], |y| \le h, h \ll l\}$$

the solution of the system of dynamic equations of plane deformation of the theory of elasticity

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = \rho\frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = \rho\frac{\partial^2 \upsilon}{\partial t^2}$$
$$\frac{\partial u}{\partial x} = \frac{1-\nu^2}{E}\left[\sigma_{xx} - \frac{\nu}{1-\nu}\sigma_{yy}\right], \quad \frac{\partial \upsilon}{\partial y} = \frac{1-\nu^2}{E}\left[\sigma_{yy} - \frac{\nu}{1-\nu}\sigma_{xx}\right]$$
$$\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x} = \frac{1}{G}\sigma_{xy} \tag{1.1}$$

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which satisfy the boundary conditions

$$\sigma_{xy}(x,\pm h) = \pm X^{\pm}(\xi)\exp(i\omega t),\quad \sigma_{yy}(x,\pm h) = \pm Y^{\pm}(\xi)\exp(i\omega t);\quad \xi = x/l \tag{1.2}$$

and the conditions on the end sections $x = 0$, l (the attachment conditions), which we will not specify for the moment. The functions $X^{\pm}(\xi)$, $Y^{\pm}(\xi)$ are assumed to be known, and ω is the frequency of the forcing action.

To determine the components $\sigma_{\alpha\beta}$ of the stress tensor components and u and υ of the displacement vector, we will seek a solution of system (1.1) in the form

$$\begin{gathered}\sigma_{\alpha\beta}(x,y,t) = \sigma_{jk}(x,y)\exp(i\omega t),\quad \alpha,\beta = x,y;\quad j,k = 1,2\\ (u,\upsilon) = (u_x(x,y),u_y(x,y))\exp(i\omega t)\end{gathered} \tag{1.3}$$

and we will change to dimensionless coordinates and displacements

$$\xi = x/l,\quad \zeta = y/h,\quad U = u_x/l,\quad V = u_y/l \tag{1.4}$$

As a result, system (1.1) takes the form

$$\begin{gathered}\frac{\partial\sigma_{11}}{\partial\xi} + \varepsilon^{-1}\frac{\partial\sigma_{12}}{\partial\zeta} + \varepsilon^{-2}\omega_*^2 U = 0,\quad \frac{\partial\sigma_{12}}{\partial\xi} + \varepsilon^{-1}\frac{\partial\sigma_{22}}{\partial\zeta} + \varepsilon^{-2}\omega_*^2 V = 0\\ \frac{\partial U}{\partial\xi} = \frac{1-\nu^2}{E}\left[\sigma_{11} - \frac{\nu}{1-\nu}\sigma_{22}\right],\quad \varepsilon^{-1}\frac{\partial V}{\partial\zeta} = \frac{1-\nu^2}{E}\left[\sigma_{22} - \frac{\nu}{1-\nu}\sigma_{11}\right]\\ \varepsilon^{-1}\frac{\partial U}{\partial\zeta} + \frac{\partial V}{\partial\xi} = \frac{1}{G}\sigma_{12};\quad \omega_*^2 = \rho h^2\omega^2,\quad \xi = \frac{h}{l}\end{gathered} \tag{1.5}$$

The solution of the singularly perturbed system (1.5) consists of the solution of the inner problem and solutions for the boundary layers, localized in the region of the ends $x = 0$, l. We will seek a solution of the inner problem in the form

$$\sigma_{jk}^{\text{int}} = \varepsilon^{-1+s}\sigma_{jk}^{(s)}(\xi,\zeta),\quad (U^{\text{int}},V^{\text{int}}) = \varepsilon^s(U^{(s)},V^{(s)});\quad s = \overline{0,N} \tag{1.6}$$

The notation $s = \overline{0,N}$ denotes that summation over the dummy index s is carried out from $s = 0$ to $s = N$.

Substituting expressions (1.6) into system (2.5), we obtain the following system for determining $\sigma_{jk}^{(s)}$, $U^{(s)}$, $V^{(s)}$

$$\begin{gathered}\frac{\partial\sigma_{11}^{(s-1)}}{\partial\xi} + \frac{\partial\sigma_{12}^{(s)}}{\partial\zeta} + \omega_*^2 U^{(s)} = 0,\quad \frac{\partial\sigma_{12}^{(s-1)}}{\partial\xi} + \frac{\partial\sigma_{22}^{(s)}}{\partial\zeta} + \omega_*^2 V^{(s)} = 0\\ \frac{\partial U^{(s-1)}}{\partial\xi} = \frac{1-\nu^2}{E}\left[\sigma_{11}^{(s)} - \frac{\nu}{1-\nu}\sigma_{22}^{(s)}\right],\quad \frac{\partial V^{(s)}}{\partial\zeta} = \frac{1-\nu^2}{E}\left[\sigma_{22}^{(s)} - \frac{\nu}{1-\nu}\sigma_{11}^{(s)}\right]\\ \frac{\partial V^{(s-1)}}{\partial\xi} + \frac{\partial U^{(s)}}{\partial\zeta} = \frac{1}{G}\sigma_{12}^{(s)}\end{gathered} \tag{1.7}$$

System (1.7) has been derived taking into account the fact that the inertial terms play a decisive role from the very beginning and, consequently, must be present in the equations for the initial approximation, i.e., a steady vibrational process occurs. This sets limitations on the possible values of ω_*. In this case the value of ω_* must not be too small or too large compared with unity. We will show that, for example, for low-frequency vibrations the asymptotic form (1.6) is not applicable and it is necessary to obtain a different asymptotic form

From system (1.7) the stresses can be expressed in terms of the displacements

$$\begin{gathered}\sigma_{12}^{(s)} = G\frac{\partial U^{(s)}}{\partial\zeta} + G\frac{\partial V^{(s-1)}}{\partial\xi},\quad \sigma_{11}^{(s)} = \delta_1\left[\frac{\nu}{1-\nu}\frac{\partial V^{(s)}}{\partial\zeta} + \frac{\partial U^{(s-1)}}{\partial\xi}\right]\\ \sigma_{22}^{(s)} = \delta_1\left[\frac{\partial V^{(s)}}{\partial\zeta} + \frac{\nu}{1-\nu}\frac{\partial U^{(s-1)}}{\partial\xi}\right];\quad \delta_1 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\end{gathered} \tag{1.8}$$

The latter are found from the equations

$$\frac{\partial^2 U^{(s)}}{\partial\zeta^2} + \frac{\omega_*^2}{G}U^{(s)} = R_u^{(s)},\quad R_u^{(s)} = -\frac{1}{G}\left[\frac{\partial\sigma_{11}^{(s-1)}}{\partial\xi} + G\frac{\partial^2 V^{(s-1)}}{\partial\xi\partial\zeta}\right]$$

$$\frac{\partial^2 V^{(s)}}{\partial\zeta^2} + \frac{\omega_*^2}{\delta_1}V^{(s)} = R_\upsilon^{(s)},\quad R_\upsilon^{(s)} = -\left[\frac{1}{\delta_1}\frac{\partial\sigma_{12}^{(s-1)}}{\partial\xi} + \frac{\nu}{1-\nu}\frac{\partial^2 U^{(s-1)}}{\partial\xi\partial\zeta}\right]$$

and have the form

$$U^{(s)} = C_{1u}^{(s)}(\xi)\sin\alpha\zeta + C_{2u}^{(s)}(\xi)\cos\alpha\zeta + \bar{u}^{(s)}(\xi,\zeta), \quad \alpha = \frac{\omega_*}{\sqrt{G}} = h\omega\sqrt{\frac{\rho}{G}}$$

$$V^{(s)} = C_{1\upsilon}^{(s)}(\xi)\sin\beta\zeta + C_{2\upsilon}^{(s)}(\xi)\cos\beta\zeta + \bar{\upsilon}^{(s)}(\xi,\zeta), \quad \beta = h\omega\sqrt{\frac{\rho}{\delta_1}} \tag{1.9}$$

where $\bar{u}^{(s)}$, $\bar{v}^{(s)}$ are particular solutions of the same equations.

Using formulae (1.3), (1.4), (1.6), (1.8) and (1.9) and satisfying conditions (1.2), we determine the unknown functions $C_{ju}^{(s)}$, $C_{jv}^{(s)}$ in solution (1.9):

$$C_{1u}^{(s)} = \frac{1}{2\cos\alpha}(f_u^{+(s)} + f_u^{-(s)}), \quad C_{2u}^{(s)} = \frac{1}{2\sin\alpha}(f_u^{-(s)} - f_u^{+(s)})$$

$$f_u^{\pm(s)} = \frac{1}{G\alpha}\sigma_{12}^{\pm(s)} - \frac{1}{\alpha}\left(\frac{\partial\bar{u}^{(s)}}{\partial\zeta} + \frac{\partial V^{(s-1)}}{\partial\xi}\right)_{\zeta=\pm1}, \quad \sigma_{12}^{\pm(0)} = \pm\varepsilon X^{\pm}, \quad \sigma_{12}^{\pm(s)} = 0, \quad s \neq 0$$

$$C_{1\upsilon}^{(s)} = \frac{1}{2\cos\beta}(f_\upsilon^{+(s)} + f_\upsilon^{-(s)}), \quad C_{2\upsilon}^{(s)} = \frac{1}{2\sin\beta}(f_\upsilon^{-(s)} - f_\upsilon^{+(s)})$$

$$f_\upsilon^{\pm(s)} = \frac{1}{\delta_1\beta}\sigma_{22}^{\pm(s)} - \frac{1}{\beta}\left(\frac{\partial\bar{\upsilon}^{(s)}}{\partial\zeta} + \frac{\nu}{1-\nu}\frac{\partial U^{(s-1)}}{\partial\xi}\right)_{\zeta=\pm1}, \quad \sigma_{22}^{\pm(0)} = \pm\varepsilon Y^{\pm}, \quad \sigma_{22}^{\pm(s)} = 0, \quad s \neq 0 \tag{1.10}$$

and, from formulae (1.8) and (1.9), we obtain the final solution of the inner problem

$$U^{(s)} = \frac{1}{\sin2\alpha}[-f_u^{+(s)}\cos(1+\zeta)\alpha + f_u^{-(s)}\cos(1-\zeta)\alpha] + \bar{u}^{(s)}(\xi,\zeta)$$

$$\sigma_{12}^{(s)} = \frac{G\alpha}{\sin2\alpha}[f_u^{+(s)}\sin(1+\zeta)\alpha + f_u^{-(s)}\sin(1-\zeta)\alpha] + G\left(\frac{\partial\bar{u}^{(s)}}{\partial\zeta} + \frac{\partial V^{(s-1)}}{\partial\xi}\right)$$

$$V^{(s)} = \frac{1}{\sin2\beta}[-f_\upsilon^{+(s)}\cos(1+\zeta)\beta + f_\upsilon^{-(s)}\cos(1-\zeta)\beta] + \bar{\upsilon}^{(s)}(\xi,\zeta)$$

$$\sigma_{11}^{(s)} = \frac{\beta\delta_1}{\sin2\beta}\left[\frac{\nu}{1-\nu}(f_\upsilon^{+(s)}\sin(1+\zeta)\beta + f_\upsilon^{-(s)}\sin(1-\zeta)\beta)\right] + \delta_1\left(\frac{\nu}{1-\nu}\frac{\partial\bar{\upsilon}^{(s)}}{\partial\zeta} + \frac{U^{(s-1)}}{\partial\xi}\right)$$

$$\sigma_{22}^{(s)} = \frac{\beta\delta_1}{\sin2\beta}[f_\upsilon^{+(s)}\sin(1+\zeta)\beta + f_\upsilon^{-(s)}\sin(1-\zeta)\beta] + \delta_1\left(\frac{\partial\bar{\upsilon}^{(s)}}{\partial\zeta} + \frac{\nu}{1-\nu}\frac{\partial U^{(s-1)}}{\partial\xi}\right) \tag{1.11}$$

Solution (1.11) will be finite if $\sin 2\alpha \neq 0$, $\sin 2\beta \neq 0$, i.e., when

$$\alpha \neq \frac{\pi n}{2} \quad \text{or} \quad \omega \neq \frac{\pi n}{2h}\sqrt{\frac{G}{\rho}} = \frac{\pi n}{2h}\upsilon_s$$

$$\beta \neq \frac{\pi k}{2} \quad \text{or} \quad \omega \neq \frac{\pi k}{2h}\sqrt{\frac{\delta_1}{\rho}} = \frac{\pi k}{2h}\upsilon_p; \quad n, k \in N \tag{1.12}$$

where

$$\upsilon_s = \sqrt{\frac{G}{\rho}}, \quad \upsilon_p = \sqrt{\frac{\delta_1}{\rho}} = \sqrt{\frac{E}{\rho}\frac{1-\nu}{(1+\nu)(1-2\nu)}}$$

(v_s and v_p are the propagation velocities of shear and longitudinal waves, known in the theory of elasticity and seismology). When conditions (1.12) are not satisfied resonance occurs.

Note that the values of ω, for which conditions (1.12) are not satisfied, agree with the principal values of the frequencies of natural vibrations.[7,9] Knowing the frequency ω of the external action, for example, seismological action, using formula (1.12) one can always choose the parameters of the strip-beam which can serve as a foundation of a building in order to avoid resonance.

The solution of the dynamic first inner problem (1.3), (1.4), (1.6), (1.11) has a number of characteristic features compared with the solution of the static first boundary-value problem of the theory of elasticity:[1,2] In the dynamic problem the asymptotic form (1.6) is fundamentally different from the asymptotic form in the static problem:[1,2]

$$\sigma_{xx}^{\text{int}} = \varepsilon^{-2+s}\sigma_{xx}^{(s)}(\xi,\zeta), \quad \sigma_{xy}^{\text{int}} = \varepsilon^{-1+s}\sigma_{xy}^{(s)}(\xi,\zeta), \quad \sigma_{yy}^{\text{int}} = \varepsilon^{s}\sigma_{yy}^{(s)}(\xi,\zeta)$$

$$U^{\text{int}} = \varepsilon^{-2+s}U^{(s)}(\xi,\zeta), \quad V^{\text{int}} = \varepsilon^{-3+s}V^{(s)}(\xi,\zeta); \quad s = \overline{0, N} \tag{1.13}$$

i.e., the form of the stress-strain state is radically changed; the solution of the dynamic inner problem, as follows from formulae (1.9)–(1.11), is completely determined from the conditions when $y = \pm h$, whereas in the static problem the conditions on the end sections $x = 0, l$ affect the solution of the static problem; in the dynamic problem the conditions when $x = 0, l$ only affect the solution in the boundary layer.

It follows from formulae (1.9)–(1.11) that two types of waves arise in the rectangle – shear waves with a characteristic amplitude $U^{(s)}$ and longitudinal waves with a characteristic amplitude $V^{(s)}$. When $s = 0$ these waves are mutually independent, and when $s \geq 1$ they interfere and it is of the order of ε. When $X^{\pm} = \text{const}$, $Y^{\pm} = \text{const}$ the iteration process is terminated at the initial approximation and the following exact solution of the inner problem is obtained:

$$
\begin{aligned}
u^{\text{int}} &= -\frac{h}{G\alpha\sin 2\alpha}[X^+\cos(1+\zeta)\alpha + X^-\cos(1-\zeta)\alpha]\exp(i\omega t) \\
\sigma_{xy}^{\text{int}} &= \frac{1}{\sin 2\alpha}[X^+\sin(1+\zeta)\alpha - X^-\sin(1-\zeta)\alpha]\exp(i\omega t) \\
\upsilon^{\text{int}} &= -\frac{h}{\beta\delta_1\sin 2\beta}[Y^+\cos(1+\zeta)\beta + Y^-\cos(1-\zeta)\beta]\exp(i\omega t) \\
\sigma_{xx}^{\text{int}} &= \frac{\nu}{1-\nu}\sigma_{yy}^{\text{int}} = \frac{\nu}{1-\nu}\frac{1}{\sin 2\beta}[Y^+\sin(1+\zeta)\beta - Y^-\sin(1-\zeta)\beta]\exp(i\omega t)
\end{aligned}
\tag{1.14}
$$

The iteration process is terminated and an exact solution of the inner problem is obtained when the functions $X^{\pm}(\xi)$, $Y^{\pm}(\xi)$ are polynomials.

The solution of the inner problem (1.3), (1.6), (1.11) or (1.13), (1.6), (1.14), generally speaking, will not satisfy the boundary conditions on the side surface (the ends) of the rectangle. To eliminate the discrepancy it is necessary to have a solution in the boundary layer and to match both solutions using the conditions on the side surface.

2. The solution in the boundary layer and its matching with the solution of the inner problem

In order to construct a solution in the boundary layer, localized in the region of the end $x = 0$, we will introduce the new replacement of variable[2,11,12] into system of equations (1.5), we will give the quantities the subscript b, and we will again seek a solution of the system obtained in the form

$$
\sigma_{jkb} = \varepsilon^{-1+s}\bar{\sigma}_{jkb}^{(s)}(\gamma,\zeta), \quad (U_b, V_b) = \varepsilon^s(\bar{u}_b^{(s)}(\gamma,\zeta), \bar{\upsilon}_b^{(s)}(\eta,\zeta)); \quad s = \overline{0, N}
\tag{2.1}
$$

To determine the coefficients of representation (2.1) we obtain a system similar in structure to system (1.7). Its solution we will sought in the form

$$
\bar{R}_b^{(s)} = R_b^{(s)}(\zeta)\exp(-\lambda\gamma)
\tag{2.2}
$$

We obtain

$$
\begin{aligned}
\sigma_{12b}^{(s)} &= G\left(\frac{du_b^{(s)}}{\partial\zeta} - \lambda\upsilon_b^{(s)}\right) \\
\sigma_{11b}^{(s)} &= \delta_1\left[-\lambda u_b^{(s)} + \frac{\nu}{1-\nu}\frac{d\upsilon_b^{(s)}}{d\zeta}\right], \quad \sigma_{22b}^{(s)} = \delta_1\left[\frac{d\upsilon_b^{(s)}}{d\zeta} - \lambda\frac{\nu}{1-\nu}u_b^{(s)}\right] \\
(\lambda^2 G + \omega_*^2)\upsilon_b^{(s)} &= -\frac{1}{\lambda}\left[2G(1-\nu)\frac{d^3u_b^{(s)}}{d\zeta^3} + (G(3-2\nu)\lambda^2 + 2(1-\nu)\omega_*^2)\frac{du_b^{(s)}}{d\zeta}\right]
\end{aligned}
\tag{2.3}
$$

The function $u_b^{(s)}$ is found from the equation

$$
\begin{aligned}
&\frac{d^4u_b^{(s)}}{d\zeta^4} + \left[2 + \frac{3-4\nu}{2G(1-\nu)}\left(\frac{\omega_*}{\lambda}\right)^2\right]\lambda^2\frac{d^2u_b^{(s)}}{d\zeta^2} + \\
&+\frac{1-2\nu}{2(1-\nu)}\left[1 + \frac{1}{G}\left(\frac{\omega_*}{\lambda}\right)^2\right]\left[\frac{2(1-\nu)}{1-2\nu} + \frac{1}{G}\left(\frac{\omega_*}{\lambda}\right)^2\right]\lambda^4u_b^{(s)} = 0
\end{aligned}
\tag{2.4}
$$

Having the solution of Eq. (2.4), the stresses can be found from the formulae

$$\sigma_{12b}^{(s)} = \frac{G}{1+m}\left[\frac{2(1-\nu)}{\lambda^2}\frac{d^3u_b^{(s)}}{d\zeta^3} + (1+(3-2\nu)(1+m))\frac{du_b^{(s)}}{d\zeta}\right]$$

$$\sigma_{11b}^{(s)} = \frac{2\nu G}{\lambda}\frac{d^2u_b^{(s)}}{d\zeta^2} + 2G\lambda[-1+\nu(1+m)]u_b^{(s)}$$

$$\sigma_{22b}^{(s)} = \frac{2G(1-\nu)}{\lambda}\frac{d^2u_b^{(s)}}{d\zeta^2} + 2G\lambda[1+(1-\nu)(1+m)]u_b^{(s)}; \quad m = \frac{1}{G}\left(\frac{\omega_*}{\lambda}\right)^2 \tag{2.5}$$

The solution of Eq. (2.4) has the form

$$u_b^{(s)} = C_1^{(s)}\sin\beta_1\zeta + C_2^{(s)}\cos\beta_1\zeta + C_3^{(s)}\sin\beta_2\zeta + C_4^{(s)}\cos\beta_2\zeta$$

$$\beta_1 = \lambda\sqrt{2+\frac{1-2\nu}{1-\nu}m}, \quad \beta_2 = \lambda\sqrt{2(1+m)} \tag{2.6}$$

The constants $C_i^{(s)}$ in solution (2.6) must be obtained from the conditions with $y=\pm h$ ($\xi=\pm 1$). Since the inhomogeneous conditions (1.2) have already been satisfied, the solution in the boundary layer must satisfy the homogeneous conditions

$$\sigma_{12b}(\zeta=\pm 1) = 0, \quad \sigma_{22b}(\zeta=\pm 1) = 0 \tag{2.7}$$

From formulae (2.5), calculating $\sigma_{12b}^{(s)}$, $\sigma_{22b}^{(s)}$ and satisfying conditions (2.7), we obtain a system of four homogeneous algebraic equations in the four unknowns $C_i^{(s)}$, which can be split into two subsystems in $C_1^{(s)}$, $C_3^{(s)}$ and in $C_2^{(s)}$, $C_4^{(s)}$:

$$C_1^{(s)}D_{11}\beta_1\cos\beta_1 + C_3^{(s)}D_{21}\beta_2\cos\beta_2 = 0, \quad C_1^{(s)}D_{12}\sin\beta_1 + C_3^{(s)}D_{22}\sin\beta_2 = 0 \tag{2.8}$$

$$C_2^{(s)}D_{11}\beta_1\sin\beta_1 + C_4^{(s)}D_{21}\beta_2\sin\beta_2 = 0, \quad C_2^{(s)}D_{12}\cos\beta_1 + C_4^{(s)}D_{22}\cos\beta_2 = 0 \tag{2.9}$$

where

$$D_{jk} = (3-k)(1-\nu)\beta_j^2 - \lambda^2 b_k, \quad j,k = 1,2$$

$$b_1 = 1+(3-2\nu)(1+m), \quad b_2 = 1+(1-\nu)(1+m)$$

System (2.8) will have a non-zero solution if its determinant is equal to zero. As a result, we obtain the following equation for determining λ

$$\Delta_- = 0 \tag{2.10}$$

We have used the following notation here

$$\Delta_\pm = (\beta_1-\beta_2)(d_1+d_2)\sin(\beta_1+\beta_2) \pm (\beta_1+\beta_2)(d_1-d_2)\sin(\beta_1-\beta_2) = 0$$

$$\beta_1 = \sqrt{2\lambda^2 + \frac{1-2\nu}{1-\nu}\frac{\omega_*^2}{G}}, \quad \beta_2 = \sqrt{2\left(\lambda^2 + \frac{\omega_*^2}{G}\right)}$$

$$d_1 = 2(1-\nu)^2\beta_1^2\beta_2^2 - 2(1-\nu)\bar{b}_2(\beta_1^2+\beta_2^2) + \bar{b}_1\bar{b}_2 =$$

$$= 2\nu^2\lambda^4 + (2-3\nu+4\nu^2)\lambda^2\frac{\omega_*^2}{G} + (1-\nu)(1-2\nu)\frac{\omega_*^4}{G^2}$$

$$d_2 = (1-\nu)\beta_1\beta_2(\bar{b}_1 - 2\bar{b}_2) = (1-\nu)\beta_1\beta_2\frac{\omega_*^2}{G}$$

$$\bar{b}_1 = \lambda^2 + (3-2\nu)\left(\lambda^2 + \frac{\omega_*^2}{G}\right), \quad \bar{b}_2 = \lambda^2 + (1-\nu)\left(\lambda^2 + \frac{\omega_*^2}{G}\right)$$

The following transcendental equation corresponds to system (2.9)

$$\Delta_+ = 0 \tag{2.11}$$

If λ is the root of Eq. (2.10), it will be the root of Eq. (2.11) and vice versa. Consequently, the non-zero solution of system (2.8) and the identically zero solution ($C_2^{(s)} = C_4^{(s)} \equiv 0$) of system (2.9) will correspond to Eq. (2.10). If λ is the root of Eq. (2.11), $C_1^{(s)} = C_3^{(s)} \equiv 0$ will correspond to it.

Table 1

ν	$\omega_*^2/G = 0.01$	0.1	1	10
$\Delta_- = 0$				
	0.6563	0.9310	0.9611	$2.9585 + 0.2299i$
	$2.6398 + 0.9638i$	$25737 + 0.8284i$	2.2152	5.0143
0.2	$4.9115 + 1.1802i$	$4.8889 + 1.1366i$	2.7387	6.4134
	$7.1541 + 1.3116i$	$7.1431 + 1.2899i$	$4.7836 + 0.8266i$	6.7901
	$9.3877 + 1.4068i$	$9.3810 + 1.3938i$	$7.0669 + 1.1138i$	$8.9695 + 0.7024i$
	0.5393	0.8155	0.9414	2.7709
	$2.6458 + 0.9733i$	$2.6058 + 0.9217i$	$2.4394 + 0.5029i$	3.4588
0.3	$4.9129 + 1.1833i$	$4.9001 + 1.1668i$	$4.8017 + 1.0209i$	4.6798
	$7.1546 + 1.3132i$	$7.1479 + 1.3050i$	$7.0873 + 1.2296i$	$6.6276 + 0.7291i$
	$9.3880 + 1.4077i$	$9.3836 + 1.4029i$	$9.3416 + 1.3569i$	$8.9933 + 1.0322i$
	0.4575	0.7143	0.8843	2.2583
	$2.6479 + 0.9763i$	$2.6231 + 0.9532i$	$2.4551 + 0.7302i$	$4.1815 + 0.4246i$
0.4	$4.9134 + 1.1844i$	$4.9047 + 1.1769i$	$4.8260 + 1.1059i$	$6.6808 + 0.9758i$
	$7.1549 + 1.3137i$	$7.1499 + 1.3100i$	$7.1022 + 1.2746i$	$9.0365 + 1.1970i$
	$9.3881 + 1.4080i$	$9.3847 + 1.4059i$	$9.3510 + 1.3847i$	$11.3392 + 1.3367i$
$\Delta_+ = 0$				
	$1.4504 + 0.7546i$	$1.3766 + 0.3520i$	0.1670	$1.2468 + 0.6712i$
	$3.7884 + 1.0891i$	$3.7457 + 1.0191i$	1.9880	3.9786
0.2	$6.0345 + 1.2521i$	$6.0192 + 1.2222i$	$3.6322 + 0.5714i$	4.7264
	$8.2716 + 1.3624i$	$8.2632 + 1.3459i$	$5.9288 + 0.9931i$	5.9931
	$10.5028 + 1.4462i$	$10.4973 + 1.4357i$	$801990 + 1.2066i$	$7.7970 + 0.4981i$
	$1.4726 + 0.7812i$	$1.3663 + 0.6122i$	$0.2919 + 0.2947i$	$1.4136 + 0.5390i$
	$3.7850 + 1.0891i$	$3.7644 + 1.0674i$	1.7974	3.7709
0.3	$6.0353 + 1.2542i$	$6.0268 + 1.2430i$	$3.6341 + 0.8401i$	$5.3969 + 0.4154i$
	$8.2720 + 1.3636i$	$8.2666 + 1.3574i$	$5.9504 + 1.1410i$	$7.8208 + 0.9055i$
	$10.5030 + 1.4469i$	$10.4992 + 1.4430i$	$8.2169 + 1.2994i$	$10.1525 + 1.1315i$
	$1.4805 + 0.7892i$	$1.4081 + 0.7195i$	$0.4500 + 0.0891i$	1.2429
	$3.7859 + 1.0958i$	$3.7726 + 1.0837i$	1.5441	1.9436
0.4	$6.0356 + 1.2549i$	$6.0293 + 1.2499i$	$3.6618 + 0.9691i$	3.5327
	$8.2721 + 1.3640i$	$8.2681 + 1.3612i$	$5.9695 + 1.2013i$	$5.4622 + 0.7917i$
	$10.5031 + 1.4472i$	$10.5001 + 1.4454i$	$8.2285 + 1.3343i$	$7.8681 + 1.1016i$

With condition (2.10), from system (2.8) $C_3^{(s)}$ is expressed in terms of $C_1^{(s)}$, and we have the solution

$$u_b^{(s)} = \bar{C}_1^{(s)}(E_{22}\sin\beta_2\sin\beta_1\zeta - E_{12}\sin\beta_1\sin\beta_2\zeta), \quad \bar{C}_1^{(s)} = C_1^{(s)}/(E_{22}\sin\beta_2) \tag{2.12}$$

where $\bar{C}_1^{(s)}$ is a new arbitrary constant.

The following solution corresponds to Eq. (2.11)

$$u_b^{(s)} = \bar{C}_2^{(s)}(E_{22}\cos\beta_2\cos\beta_1\zeta - E_{12}\cos\beta_1\cos\beta_2\zeta), \quad \bar{C}_2^{(s)} = C_2^{(s)}/(E_{22}\sin\beta_2) \tag{2.13}$$

We have used the following notation here

$$E_{j2} = (1-\nu)\beta_j^2 - \bar{b}_2, \quad j = 1, 2$$

Solution (2.12) corresponds to the skew-symmetric problem, while (2.13) corresponds to the symmetric problem. If λ is the root of Eq. (2.10) or (2.11), then $-\lambda$, $\bar{\lambda}$ are also roots of the same equations. In view of the property of the boundary layer, the roots with $\mathrm{Re}\,\lambda > 0$ are of interest, since only in this case will the solution be attenuating with distance from the end $x = 0$ inside the rectangular region.

We show in Table 1 values of the first five roots $\lambda_n = x_n + iy_n$ c $\mathrm{Re}\,\lambda_n > 0$ of Eq. (2.10) and Eq. (2.11) for certain values of the parameter ω_*^2/G and Poisson's ratio ν.

The solution (2.1), (2.2) can be written in the form

$$u_b = \varepsilon^s U_b^{(s)}, \quad s = \overline{0, N}, \quad U_b^{(s)} = lu_b^{(s)}(\zeta)\exp(-\lambda\gamma) \ (u, v; U, V)$$

$$\sigma_{xxb} = \varepsilon^{-1+s}\sigma_{xxb}^{(s)}, \quad s = \overline{0, N}, \quad \sigma_{xxb}^{(s)} = \sigma_{11b}^{(s)}(\zeta)\exp(-\lambda\gamma) \ (x, y; 1, 2)$$

$$\sigma_{xyb} = \varepsilon^{-1+s}\sigma_{xyb}^{(s)}, \quad s = \overline{0, N}, \quad \sigma_{xyb}^{(s)} = \sigma_{12b}^{(s)}(\zeta)\exp(-\lambda\gamma) \tag{2.14}$$

The function $u_b^{(s)}$ is found from formula (2.12) or (2.13), while $\sigma_{jkb}^{(s)}$ is found from formula (2.5).

Since $\bar{\lambda}$ corresponds to each λ, we can write solution (2.14) in such a way that only real functions occur in it. Arranging the roots λ_n of Eqs (2.10) and (2.11) in increasing order of their real part $\mathrm{Re}\,\lambda_n$, taking $\bar{C}_1^{(s)} = (A_{1n}^{(s)} - iA_{2n}^{(s)})/2$ and writing

$$U_b^{(s)} = \mathrm{Re}\,U_b^{(s)} + i\,\mathrm{Im}\,U_b^{(s)} (U, V), \quad \sigma_{xxb}^{(s)} = \mathrm{Re}\,\sigma_{xxb}^{(s)} + i\,\mathrm{Im}\,\sigma_{xxb}^{(s)} (x, y) \tag{2.15}$$

we can represent formulae (2.14) in the form

$$u_b = \varepsilon^s (A_{1n}^{(s)} \mathrm{Re} U_{bn} + A_{2n}^{(s)} \mathrm{Im} U_{bn}) \ (u, v; U, V)$$
$$\sigma_{xxb} = \varepsilon^{-1+s} (A_{1n}^{(s)} \mathrm{Re}\sigma_{xxbn} + A_{2n}^{(s)} \mathrm{Im}\sigma_{xxbn})$$
$$\sigma_{xyb} = \varepsilon^{-1+s} (A_{1n}^{(s)} \mathrm{Re}\sigma_{xybn} + A_{2n}^{(s)} \mathrm{Im}\sigma_{xybn})$$
$$s = \overline{0, N} \tag{2.16}$$

When calculating Re R_b, in formulae (2.5) and (2.14) we must take as u_b the coefficient of $\bar{C}_1^{(s)}$ or $\bar{C}_2^{(s)}$ in formulae (2.12) and (2.13), which leads to the omission in (2.16) of the superscript s on $U_b^{(s)}$, $\sigma_{jkb}^{(s)}$. The real constants $A_{1n}^{(s)}, A_{2n}^{(s)}$ must be determined during the course of matching the solution of the inner problem and the solution in the boundary layer. The boundary layer, corresponding to the edge $x = l$, is constructed in a similar way. If we count from $x = 0$, the data of this boundary layer can be obtained from the above by replacing γ by $\gamma_1 = \varepsilon^{-1} \cdots \gamma = (l - x)/h$.

We will consider the procedure for satisfying the boundary conditions for $x = 0$. Suppose we have a free edge

$$\sigma_{xx} = 0, \quad \sigma_{xy} = 0 \text{ when } x = 0 \tag{2.17}$$

The general integral of the problem has the form

$$I = Q^{\mathrm{int}} + R_b^{\mathrm{I}} + R_b^{\mathrm{II}} \tag{2.18}$$

When satisfying the conditions when $x = 0$ the effect of the boundary layer R_b^{II}, localised in the region of $x = l$, is usually neglected. This imposes a limitation on the dimensions of the rectangle: it is necessary that the following condition should be satisfied

$$1 + \exp\left(-\mathrm{Re}\lambda_1 \frac{l}{h}\right) \approx 1 \tag{2.19}$$

which in practical applications for beams and rods is always satisfied, since, as a rule, $l \geq 10$ h. Since solutions (2.12)–(2.14) in the boundary layer and the similar solutions corresponding to the edge $x = l$ are accurate for each s, it may seem that the above assumption is not necessary. However, this is hardly justified since it leads to only a small additional contribution with an excessive increase in the amount of calculation.

Taking formulae (1.3), (1.6), (2.1), (2.2), (2.5), (2.16), (2.18) and the above into account, we can write conditions (2.17) in the form

$$\bar{\sigma}_{11}^{(s)}(\xi = 0, \zeta) + (A_{1n}^{(s)} \mathrm{Re}\sigma_{xxbn} + A_{2n}^{(s)} \mathrm{Im}\sigma_{xxbn})_{\gamma = 0} = 0$$
$$\bar{\sigma}_{12}^{(s)}(\xi = 0, \zeta) + (A_{1n}^{(s)} \mathrm{Re}\sigma_{xxbn} + A_{2n}^{(s)} \mathrm{Im}\sigma_{xybn})_{\gamma = 0} = 0; \quad n = \overline{0, k} \tag{2.20}$$

where k is the number of chosen boundary functions corresponding to λ_k. In the skew-symmetric problem σ_{11}, σ_{xxb} and u_b are odd functions of ζ, and σ_{12}, v_b are even functions of ζ, and conversely in the symmetric problem. Consequently

$$\bar{\sigma}_{11}^{(s)}(0, \zeta) = \frac{\sigma_{11}^{(s)}(0, \zeta) \mp \sigma_{11}^{(s)}(0, -\zeta)}{2}, \quad \bar{\sigma}_{12}^{(s)}(0, \zeta) = \frac{\sigma_{12}^{(s)}(0, \zeta) \pm \sigma_{12}^{(s)}(0, -\zeta)}{2} \tag{2.21}$$

respectively in the skew-symmetric and symmetric problems, and $\sigma_{11}^{(s)}$, $\sigma_{12}^{(s)}$ are already known functions, calculated from formulae (1.11). The functions u_b, σ_{xxbn}, σ_{xybn} are calculated from formulae (2.12), (2.5) and (2.14) in the skew-symmetric problem and from (2.13), (2.5) and (2.14) in the symmetric problem. The determination of the unknown constants $A_{1n}^{(s)}, A_{2n}^{(s)}$ from system (2.20) can be reduced to solving a system of algebraic equations if we use the collocation method, the method of least squares or Fourier's method. Naturally, they can be calculated approximately, but with a specified accuracy.

The solution of the inner problem and the solution in the boundary layer for other boundary conditions at the ends $x = 0, l$ can be matched in a similar way. For example, for conditions of rigid clamping ($u = 0$ and $v = 0$ when $x = 0$) we have

$$\bar{u}^{(s)}(\xi = 0, \zeta) + (A_{1n}^{(s)} \mathrm{Re} U_{bn} + A_{2n}^{(s)} \mathrm{Im} U_{bn})_{\gamma = 0} = 0 \ (u, v; U, V); \quad n = \overline{0, k} \tag{2.22}$$

where

$$\bar{u}^{(s)} = l\frac{U^{(s)}(0, \zeta) \mp U^{(s)}(0, -\zeta)}{2}, \quad \bar{v}^{(s)} = l\frac{V^{(s)}(0, \zeta) \pm V^{(s)}(0, -\zeta)}{2} \tag{2.23}$$

respectively in the skew-symmetric and symmetric problems, calculated from formulae (1.11). In the skew-symmetric problem U_{bn} is calculated from formulae (2.12) and (2.14), in the symmetric problem from (2.13) and (2.14) as the coefficient of $\bar{C}_1^{(s)}$ and $\bar{C}_2^{(s)}$ respectively, and V_{bn} is calculated from formulae (2.3) and (2.14). The constant $A_{1n}^{(s)}, A_{2n}^{(s)}$ are determined by the method described for the previous case.

3. The asymptotic form of low-frequency and high-frequency vibrations

Suppose $\omega^2 = \varepsilon\Omega^2$. Then, instead of the first two equations of (1.5) we will have

$$\frac{\partial\sigma_{11}}{\partial\xi} + \varepsilon^{-1}\frac{\partial\sigma_{12}}{\partial\zeta} + \varepsilon^{-1}\Omega_*^2 U = 0, \quad \frac{\partial\sigma_{12}}{\partial\xi} + \varepsilon^{-1}\frac{\partial\sigma_{22}}{\partial\zeta} + \varepsilon^{-1}\Omega_*^2 V = 0; \quad \Omega_*^2 = \rho h^2\Omega^2 \tag{3.1}$$

The solution of the system consisting of Eqs (3.1) and the remaining equations of (1.5) will be sought in the form

$$Q^{\text{int}} = \varepsilon^{q_Q + s} Q^{(s)}(\xi, \zeta), \quad s = \overline{0, N} \tag{3.2}$$

To determine the functions $Q^{(s)}(\xi, \xi)$ we obtain a non-contradictory system only when

$$q_{\sigma_{11}} = q_{\sigma_{12}} = -2, \quad q_{\sigma_{22}} = -1, \quad q_U = -2, \quad q_V = -1 \tag{3.3}$$

i.e., for low-frequency vibrations the asymptotic form (3.2), (3.3) differs in principle from the asymptotic form (1.6). Substituting expressions (3.2) into Eqs (3.1) and the elasticity relations (1.5) and taking boundary conditions (1.2) into account we obtain the solution

$$U^{(s)} = u_0^{(s)}(\xi) + u_*^{(s)}(\xi, \zeta), \quad u_*^{(s)} = \int_0^\zeta \left(\frac{1}{G}\sigma_{12}^{(s-1)} - \frac{\partial V^{(s-2)}}{\partial\xi}\right)d\zeta$$

$$\sigma_{11}^{(s)} = \frac{E}{1-\nu^2}\frac{du_0^{(s)}}{d\xi} + \sigma_{11*}^{(s)}, \quad \sigma_{11*}^{(s)} = \frac{E}{1-\nu^2}\frac{\partial u_*^{(s)}}{\partial\xi} + \frac{\nu}{1-\nu}\sigma_{22}^{(s-1)}$$

$$V^{(s)} = \upsilon_0^{(s)}(\xi) - \frac{\nu}{1-\nu}\frac{du_0^{(s)}}{d\xi}\zeta + \upsilon_*^{(s)}(\xi, \zeta)$$

$$\upsilon_*^{(s)}(\xi, \zeta) = \frac{1-\nu^2}{E}\int_0^\zeta \left(\sigma_{22}^{(s-1)} - \frac{\nu}{1-\nu}\sigma_{11*}^{(s)}\right)d\zeta$$

$$\sigma_{12}^{(s)} = \sigma_{120}^{(s)}(\xi) - \Omega_*^2 u_0^{(2)}\zeta + \sigma_{12*}^{(s)}(\xi, \zeta), \quad \sigma_{12*}^{(s)} = -\int_0^\zeta \left(\Omega_*^2 u_*^{(s)} + \frac{\partial\sigma_{11}^{(s-1)}}{\partial\xi}\right)d\zeta$$

$$\sigma_{22}^{(s)} = \sigma_{220}^{(s)}(\xi) - \left(\Omega_*^2\upsilon_0^{(s)} + \frac{d\sigma_{120}^{(s)}}{d\xi}\right)\zeta + \frac{\Omega_*^2}{1-\nu}\frac{du_0^{(s)}}{d\xi}\frac{\zeta^2}{2} + \sigma_{22*}^{(s)}(\xi, \zeta)$$

$$\sigma_{22*}^{(s)} = -\int_0^\zeta \left(\Omega_*^2\upsilon_*^{(s)} + \frac{\partial\sigma_{12*}^{(s)}}{\partial\xi}\right)d\zeta$$

$$\sigma_{120}^{(s)} = \frac{1}{2}[X^{+(s)} - X^{-(s)} - \sigma_{12*}^{(s)}(\xi, 1) - \sigma_{12*}^{(s)}(\xi, -1)]$$

$$\sigma_{220}^{(s)} = \frac{1}{2}\left[Y^{+(s)} - Y^{-(s)} - \frac{\Omega_*^2}{1-\nu}\frac{du_0^{(s)}}{d\xi} - \sigma_{22*}^{(s)}(\xi, 1) - \sigma_{22*}^{(s)}(\xi, -1)\right]$$

$$u_0^{(s)} = \frac{1}{2\Omega_*^2}[-X^{+(s)} - X^{-(s)} - \sigma_{12*}^{(s)}(\xi, -1) + \sigma_{12*}^{(s)}(\xi, 1)]$$

$$\upsilon_0^{(s)} = \frac{1}{2\Omega_*^2}\left[-Y^{+(s)} - Y^{-(s)} - 2\frac{d\sigma_{120}^{(s)}}{d\xi} + \sigma_{22*}^{(s)}(\xi, 1) - \sigma_{22*}^{(s)}(\xi, -1)\right]$$

$$X^{\pm(0)} = \varepsilon^2 X^{\pm}, \quad Y^{\pm(0)} = \varepsilon Y^{\pm}, \quad X^{\pm(s)} = Y^{\pm(s)} = 0, \quad s \neq 0, \quad Q^{(k)} \equiv 0 \text{ for } k < 0 \tag{3.4}$$

Recurrence formulae (3.4) enable us to calculate the components of the stress tensor and the displacement vector with previously specified asymptotic accuracy. They are a ready algorithm for determining an analytical solution for an arbitrary approximation using a computer.

If $\omega = \varepsilon\Omega$, in system (3.1), instead of the factor $\varepsilon^{-1}\Omega_*^2$, we have Ω_*^2. A solution is sought in the form (3.2). A non-contradictory system is obtained for the asymptotic form

$$q_{\sigma_{11}} = -1, \quad q_{\sigma_{12}} = q_{\sigma_{22}} = 0, \quad q_u = q_\upsilon = -1 \tag{3.5}$$

The solution of the inner problem, corresponding to relations (3.2), (3.5) and (1.2) is easily written down. The boundary layer is constructed in the same way as in Section 2.

For high-frequency vibrations, corresponding, for example, to $\omega = (\sqrt{\varepsilon})^{-1}\Omega$ or $\omega = \varepsilon^{-1}\Omega$, the asymptotic form (1.3) and (1.6) is preserved, but in Eqs (1.5) we will have $\omega_*^2 = \rho h l \Omega^2$ in the first case and $\omega_*^2 = \rho l^2 \Omega^2$ in the second.

In conclusion we note that the dynamic first boundary-value problem of the theory of elasticity for plates and shells is solved by the same method.

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